**Nonlinear Equations**

So now we’ll take a look at a bunch of nonlinear equations. First we’ll do 1st order equations.

**Separation of Variables**

There are a bunch of techniques to consider. The first, most basic technique is separation of variables. If we have an equation like:



Then we can say:



Then we could hopefully do the integrals and invert the LHS for y.

**First order exact equations**

Consider equations are of the form:



We’d like to write the solution as:



We can compare our differential equation with the one that would result from our knowledge of F(x,y) by taking the derivative w/r to x.



Identification of our terms would require:



Using the equality of cross-derivatives we can see that we must have:



If this is the case, then we can partially integrate N w/r to x to get F + arbitrary function of y, and then differentiate F w/r to y and set equal to M to specify that arbitrary function up to a constant. So then we’d have the general solution to our equation. However, it isn’t usually the case cross partials are equal. So then we try to use integrating factors to make it true. So we multiply the ODE by the integrating factor I(x,y) and redo the problem.



and compare to F(x,y) = c to see we now have:



and that now we need:



This condition helps us determine I.

**Example**

Let’s try,



Can write as:



This is exact as:



So solution is f(x,y) = const., where:



and,



So our general solution is:



The initial condition requires,



So our particular solution is:



**Example**

Consider another equation. A non-linear one.



Again this isn’t exact. So let’s multiply by an integrating factor. Then we need,



If I assume I isn’t a function of y, then we have:



But then our equation violates our assumption. If I assume I isn’t a function of x, then we’d have:



But that clearly doesn’t work either. So sometimes a little creativity is required to figure out an I(x,y). Whatever.

**Example: Linear Inhomogeneous ODE**

For example, consider the ODE:



So,



But ∂M/∂x ≠ ∂N/∂y. So it isn’t exact currently, and we need an integrating factor. So we multiply through by I,



Equating cross-partials, we need:



If we assume that I is just a function of x, then we get:



So then,



Now we can see that cross partials are equal. And we can proceed with the solution. Integrating w/r to y says:



Then differentiating w/r to x says,



So then our solution is:



and finally,



which is indeed the case!

**Bernoulli equation**

These have the form,



Make the substitution u = y1-p and the new equation will be linear in u. For instance consider the formal change of variables from y to y = un. Then,



If make n – np = 1 → n = 1/(1-p) then its good. And y = u1/(1-p) is equivalent to y = y1-p.



So now we have a linear inhomogeneous equation:



**Example**

For instance, let’s do:



Then we make the substitution u = y1-n,



Transforming our equation we have:



So there. Now to solve,



Solving for y, then, we have:



Yay!

**Riccati Equations**

These are kind of like inhomogeneous Bernoulli equations. They are of the form,



There is no general solution to such equations. They can be transformed into second order linear equations in w through the substitution:



The result is:



The transformation of course goes backwards too so that every second order linear ODE can be transformed into a Ricatti equation. Sometimes these might turn out nice or known. Another possibility is a sort of reduction of order. If we can guess one solution, y1(x), no matter how trivial, then we can reduce it to Bernoulli’s equation by writing:



**Higher order exact equations**

Consider an equation of the form,



This equation is exact if it is the total differential of some function F(x,y,y′) = c. Taking the derivative we’d have:



Comparing these two then, it must be the case that:



To be exact, we must have:



Hmmm….well he says the conditions for exactitude are:



**Example**

Consider the equation:





In this case M = 1, and N = xp+y. So the conditions come out to:



So this equation is exact. Therefore we have:



So F(x,y,p) = p+xy. Remember to treat p = y′ as an independent variable. And so we have:



which can be solved. Sometimes integrating factors can be used to make an equation exact.

**Example**

For example consider,



This equation isn’t exact. M = 1, N = (1+x)p/x + (x-1)y/x2. Therefore the equation yield,



which isn’t exact. It is exact when x is small though. Nonetheless, it can be made completely exact through an integrating factor. Let’s multiply through by I(x) and see if we can solve for it. Then,



And filling this in…



This is a differential equation for I. The order can be reduced by setting I(x) = eu(x). Then we have:



And could then say u′(x) = υ(x).



which is a Clariut equation I think. Anyway. Setting υ(x) = 1 is a solution. So I(x) = ex is the integrating factor. Now that we know this we have:



and solving these equations we have:



So the left hand side can be written as:



**Example**

Consider the non-linear equation:



Then we have M = 1, N = εe-y. Filling into the conditions we have:



So this isn’t exact, except for when y is very large. Then it approaches exactness. But this is because the term goes away entirely. Is there an integrating factor. Perhaps it is a function of y, I(y). Then we’ll have:



The last equation forbids it. Perhaps I(x)? Nope. Same problem. Perhaps I(p)? No again. So clearly doesn’t do everything.